Modeling Neural Population Coordination via a Block Correlation Matrix

Yunran Chen and Surya T. Tokdar
Department of Statistical Science, Duke University, Durham, NC 27708, USA

Research Question

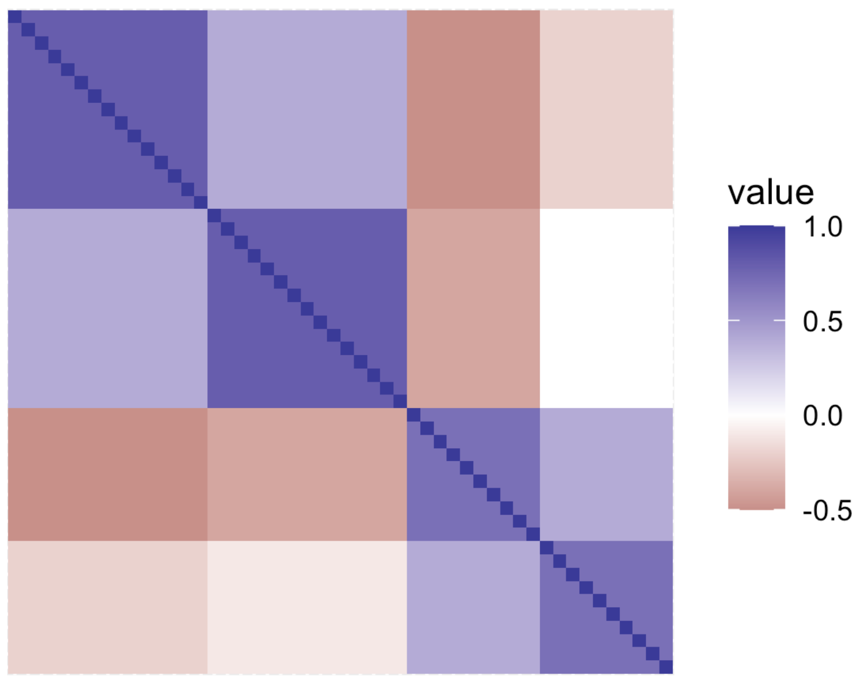
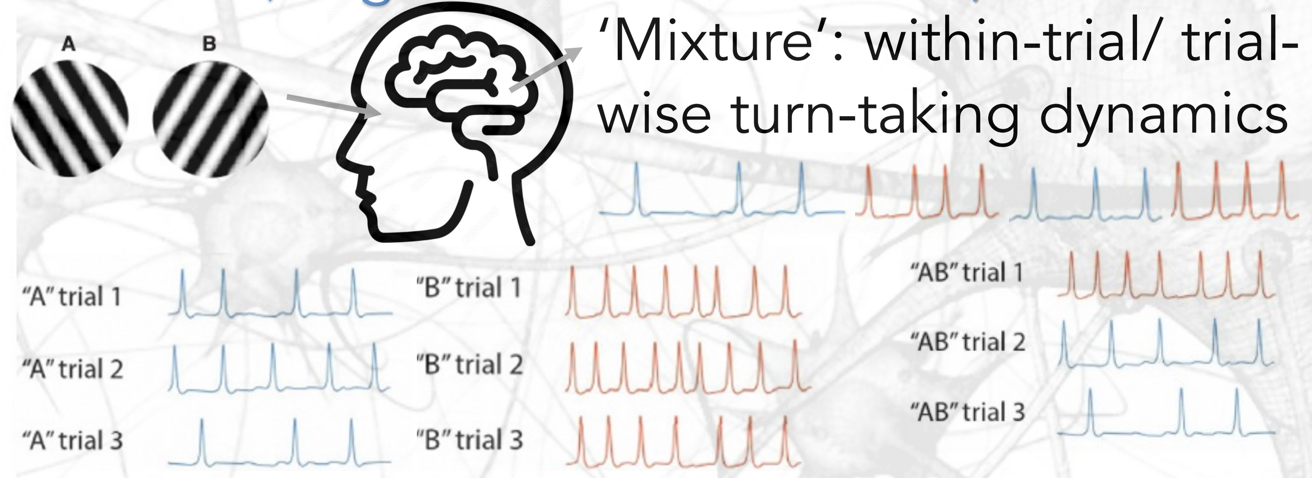


Fig. 1: An example of a block correlation matrix (50 variables 4 blocks)

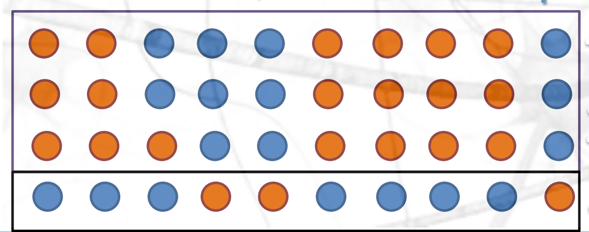
- Estimation of a block correlation matrix
- Unknown block structure: grouping w.r.t. variables
- Flexibility: off-diagonal correlation $\in$ (-1,1)
- Interpretability: model assumptions + priors
- Statistical efficiency: large p small n cases
- Computational efficiency: conjugate priors

Motivation

Phase 1 (Single neuron fluctuation)



Phase 2 (Neural Population Coordination)



Synchronously
Asynchronously
Grouping

Method: Bayesian Block Correlation Matrix Estimation

Bayesian Model: Mixture of Finite Mixtures + Canonical Representation

Groups allocation: Mixture of Finite Mixtures (MFM) (Miller and Harrison, 2018)

$\mathcal{C}$ denotes partition of $[N]$ induced by $S_1, \dots, S_N$:

$$K \sim p_K, \text{ where } p_K \text{ is a p.m.f on } \{1,2,\dots\}, \text{ where we consider } K-1 \sim \text{Pois}(1)$$

$$(\pi_1, \dots, \pi_k) \sim \text{Dir}_k(\gamma, \dots, \gamma) \text{ given } K = k$$

$$S_1, \dots, S_N \sim \pi \text{ (iid) given } \pi$$

Group allocation $S_i \in \{1, \dots, K\}$

Permuted Data: $\tilde{Y} = PY, \quad \tilde{Y} \sim N(0, \Sigma)$,

Block Covariance Matrix:

$$\Sigma = \begin{pmatrix} \Sigma_{[1,1]} & \Sigma_{[1,2]} & \dots & \Sigma_{[1,K]} \\ \Sigma_{[2,1]} & \Sigma_{[2,2]} & \dots & \Sigma_{[2,K]} \\ \vdots & & \ddots & \\ \Sigma_{[K,1]} & \Sigma_{[K,2]} & \dots & \Sigma_{[K,K]} \end{pmatrix}, \quad \Sigma_{[i,i]} = \begin{pmatrix} \sigma_i^2 & \sigma_{ii} & \dots & \sigma_{ii} \\ \sigma_{ii} & \sigma_i^2 & \dots & \sigma_{ii} \\ \vdots & & \ddots & \\ \sigma_{ii} & \sigma_{ii} & \dots & \sigma_i^2 \end{pmatrix}, \Sigma_{[i,j]} = \begin{pmatrix} \sigma_{ij} & \dots & \sigma_{ij} \\ \vdots & \ddots & \\ \sigma_{ij} & \dots & \sigma_{ij} \end{pmatrix},$$

Block Correlation Matrix maintain SAME block structure if: $\sigma_i^2 \neq \sigma_{ii}$

Canonical Representation: $\Sigma = QDQ'$,
(Archakov and Hansen, 2020)

Representatives + Replicates

Pseudo-diagonal | Orthonormal

$$D = \begin{pmatrix} A & 0 & \dots & 0 \\ 0 & \lambda_1 I_{n_1 - 1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_K I_{n_K - 1} \end{pmatrix} \quad Q = \begin{pmatrix} v_{n_1} & 0 & \dots & v_{n_1 \perp} & 0 & \dots & 0 \\ 0 & v_{n_2} & \dots & 0 & v_{n_2 \perp} & & \vdots \\ \vdots & & \ddots & \vdots & & \ddots & \\ 0 & \dots & v_{n_K} & 0 & \dots & & v_{n_K \perp} \end{pmatrix}$$

$$a_{ij} = \sigma_{ij}\sqrt{n_i n_j}, \; a_{ii} = \sigma_i^2 + (n_i - 1)\sigma_{ii} \qquad v_{n_k} = \left(\frac{1}{\sqrt{n_k}}, \dots, \frac{1}{\sqrt{n_k}}\right)'$$

$$\lambda_i = \sigma_i^2 - \sigma_{ii}. \qquad QQ' = Q'Q = I.$$

Group size (Gram-Schmidt)

Log likelihood:

$$-\frac{NJ}{2}\log 2\pi - \frac{J}{2}\log|A| - \frac{1}{2}\sum_{j=1}^{J}\eta_{j(0)}^{*'}A^{-1}\eta_{j(0)}^{*} - \frac{J}{2}\sum_{k=1}^{K}(n_k - 1)\log\lambda_k - \frac{1}{2}\sum_{j=1}^{J}\sum_{k=1}^{K}\eta_{j(k)}^{*'}\eta_{j(k)}^{*}/\lambda_k,$$

where $\eta_j^* = Q'\tilde{Y}_j$,

Conjugate Priors: $A|\mathcal{C} \sim IW(\nu, A_0)$, $\quad \lambda_k \sim (iid) IG(a_\lambda, b_\lambda)$.

Gibbs Sampler

- Initialize with one group.
- Within each iteration,
  (1) For each variable, update its group allocation based on MFM's algorithm (adaptation of 'Algorithm 3' (Neal, 2000)).
  (2) Update parameters by conjugacy.

Prior Specification

- Non-informative priors require:
  (1) invariant to group size
  (2) between-group: uniformly distributed
  (3) within-group: positive, relatively high
- Scale with group size: $\nu = K, A_{0(kk)} = n_k$, $\lambda_k^{-1} \sim Ga(20, 10)$

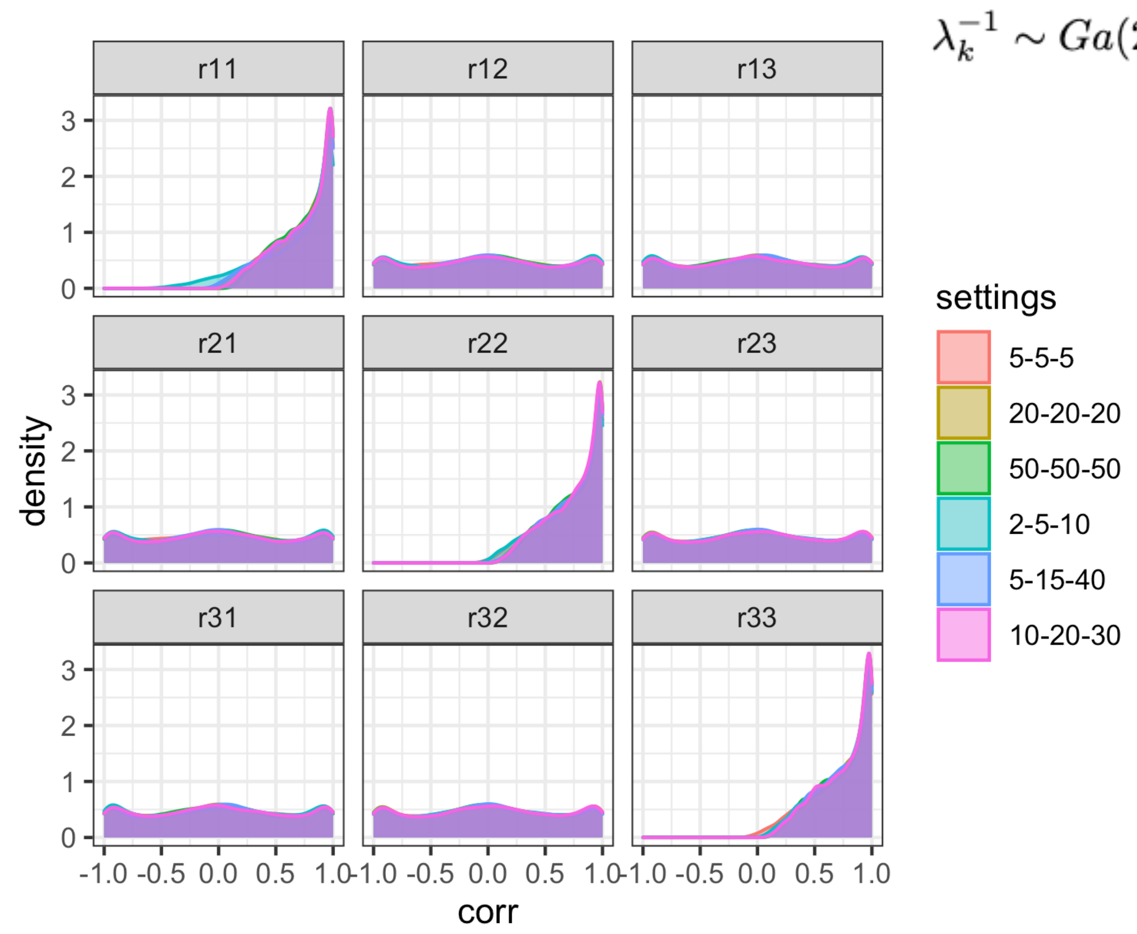
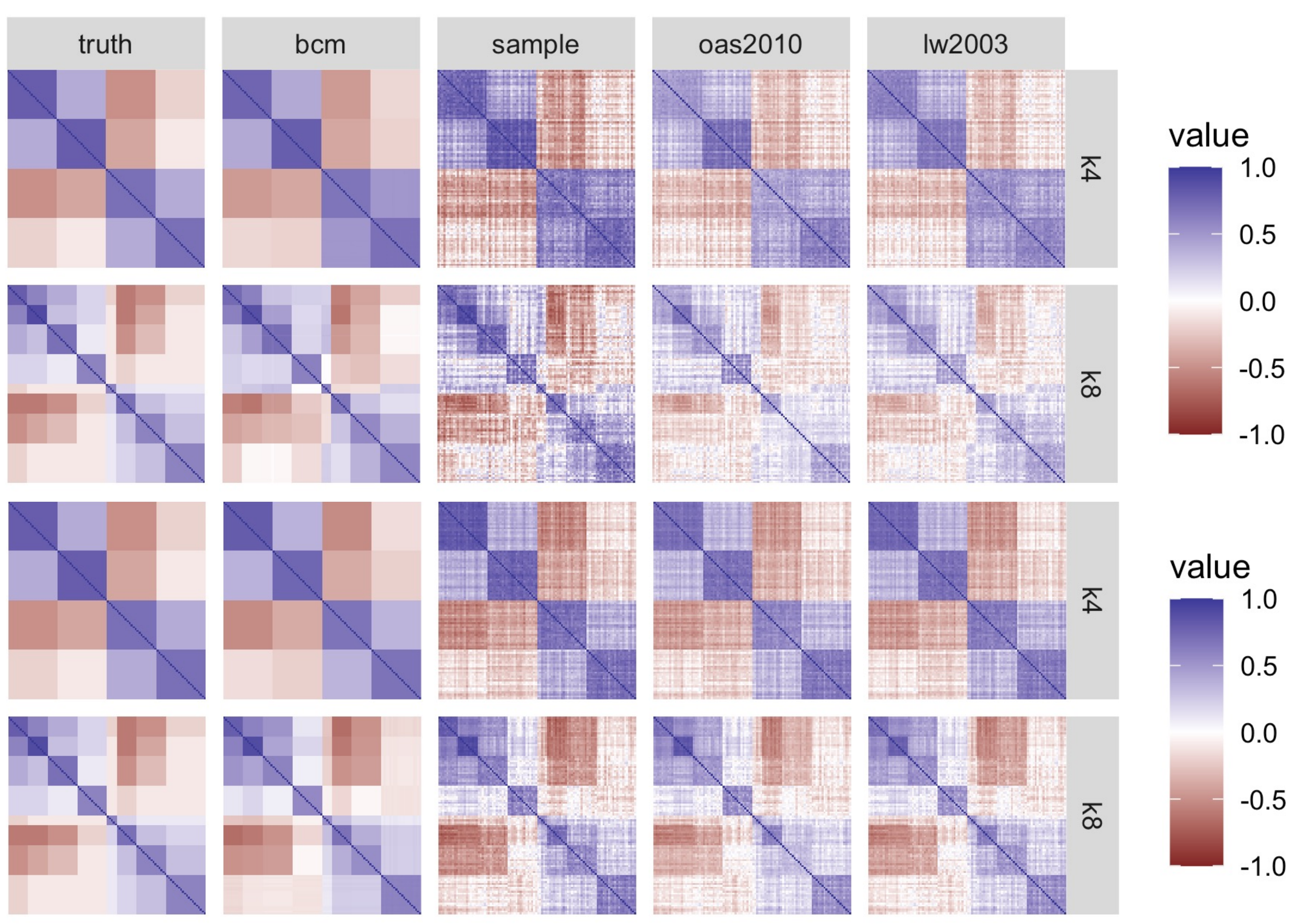


Fig. 2: Examples of induced priors for block correlation matrices under different group allocation

Numerical Experiments



- (n,p,k): (# sample, # variables, # blocks)
- Estimation accuracy: BCM outperforms other alternatives except for situations with large p/k ratio
- Grouping: BCM recovers the true block structure in a decent way (smoothing/denoise) even under small n large p cases.

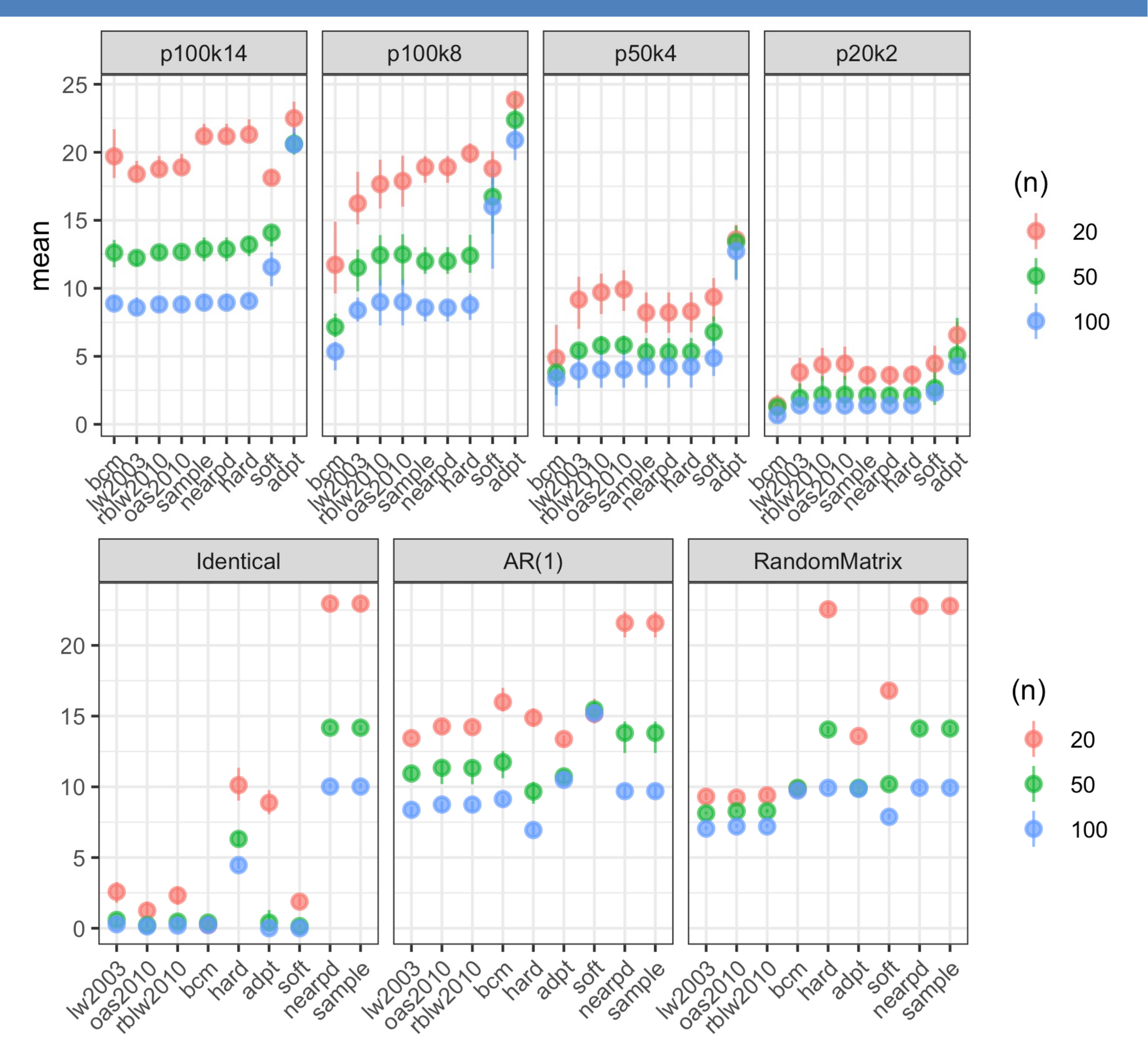


Fig. 3 (Left): Comparison between estimators when p=100, n=20 (up) or 50 (down), k=4 or 8

Fig. 4 (Right): Frobenius distance between estimators and truth under different (n,p,k) combinations